Preprint typeset in JHEP style. - HYPER VERSION

hep-th/0007067
HUTP-00/A025
LPTENS-00-30
LPTHE-00-29

Nahm's equations, $\mathcal{N} = 1^*$ domain walls, and D-strings in $AdS_5 \times S_5$

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Abstract: We consider the following two problems: classical domain walls in the $\mathcal{N} = 1^*$ mass deformation of the maximally supersymmetric Yang Mills theory, and D-strings as external magnetic sources in the context of the AdS/CFT correspondence. We show that they are both described by Nahm's equations with unconventional boundary conditions, and analyze the relevant moduli space of solutions. We argue that general 'fuzzy sphere' configurations of D-strings in AdS_5 correspond to Wilson-'t Hooft lines in higher representations of the dual $SU(n)$ gauge theory.

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1. Introduction, summary and directory

The maximally supersymmetric ($\mathcal{N} = 4$ in four dimensions) Yang Mills theory has an interesting deformation that preserves $\mathcal{N} = 1$ supersymmetry while lifting all its flat directions [1]. This theory (dubbed '$\mathcal{N} = 1^*$') has a rich phase structure, studied both with duality and instanton techniques [2, 3, 4, 5], and more recently in the context of the AdS/CFT correspondence [6, 7, 8]. A particularly intriguing feature of the theory is that it has a large number of disconnected, supersymmetric ground states. For a gauge group $SU(N)$ classical vacua are labelled by the N-dimensional representations ρ of $SU(2)$. These are determined uniquely by the dimensions of the irreducible blocks, i.e. by a partition of N in positive integers. The number of inequivalent vacua grows therefore as $\sim e^{\sqrt{N}}$ when N is large.

One of the goals of the present work is to study the existence and moduli space of classical BPS domain walls interpolating between any two of these ground states.[2] The problem has been discussed in the dual supergravity description of the $\mathcal{N} = 1^*$ theory in [5, 7, 8], with a particular emphasis on walls separating the Higgs from

[2] For earlier work on BPS walls in globally supersymmetric theories see [9, 10, 11, 12, 13, 14].

the (oblique) confining phases. Here we will consider the general situation but in the opposite, classical gauge-theory limit. Note that the existence of domain walls interpolating between a pair of discrete degenerate vacua at infinity is not a priori guaranteed on topological grounds: it could , for instance, be energetically favorable for an interpolating configuration to decompose into a pair of mutually-repelling walls with a different ground state in the middle.

What renders our problem tractable is the observation that classical BPS domain walls are described by solutions to the much studied Nahm equations. These arise in a number of situations related to monopole physics, and with a variety of boundary conditions [15, 16, 17, 18]. The boundary conditions relevant for us turn out to be precisely the ones considered previously by Kronheimer [18] in his study of SO(3)-invariant antiself-dual connections on $S^3 \times \Re \simeq S^4$. Building on his results we will bring out the following points:

(a) Supersymmetric walls exist for each pair (ρ_-, ρ_+) of vacua such that the nilpotent orbit associated to the representation ρ_+ is contained in the closure of the nilpotent orbit associated to ρ_-. This condition defines a *partial ordering* $(\rho_+ < \rho_-)$ between $SU(2)$ representations, which is easily read off from Young tableaux as illustrated in Figure 3. In particular, for a BPS domain wall to exist, it is necessary that the vacuum with the higher superpotential does not contain more irreducible blocks than the one with lower superpotential, and that the size of the biggest block does not increase as the superpotential decreases.

(b) The moduli space of domain walls is a singular hyperkähler manifold, whose dimension can be computed by Morse theory. The dimension satisfies an additivity rule displayed in Equation (4.26).

(c) This additivity of the number of massless modes suggests that all BPS walls can be decomposed into some 'elementary constituents' that interpolate between pairs of ordered neighbouring ground states. These elementary walls have no moduli other than those dictated by the gauge invariance and global R-symmetry of the problem.

These facts can be verified readily in the tables provided in section 4, which display the moduli spaces of domain walls of the $SU(N)$ $\mathcal{N} = 1^*$ theories, for all $N \leq 6$. We emphasize that our study is purely classical: it uses the tree-level superpotential and Kähler potential, and does not take into account the splitting of certain classical vacua into several oblique-confinement states. Which classical BPS walls are preserved at the full quantum level, is thus an open question that requires further investigation, possibly along the lines of [12, 19]. Note however that since the dimension of the moduli space is given, as we will argue, by a Morse index, we expect it to be at least robust under small deformations of the Kähler potential.

One of the reasons for studying the $\mathcal{N} = 1^*$ theory, apart from its possible relevance to pure QCD, is its relation to the dielectric effect [20], whereby Dp-branes in a Ramond electric background expand into D$(p+2)$-branes with the topology of a 'fuzzy' sphere. The gravity dual of the $\mathcal{N} = 1^*$ theory exhibits a supersymmetric version of this dielectric effect: the Higgs/Coulomb vacua are described by 'fuzzy' D5 branes embedded in the $AdS_5 \times S^5$ geometry [7, 8] (for other manifestations of this effect, see for example [21, 22, 23, 24, 25, 26]). The domain walls separating the Higgs from confining vacua can be described in this picture by 5-brane junctions [7]. We expect a similar interpretation for the more general domain walls, but we will not pursue this issue here any further.

We will, instead, consider another stringy situation for which the same equations apply, namely the problem of D-strings stretching out radially in $AdS_5 \times S^5$. D-strings ending on separated D3-branes in flat spacetime are described by the Nahm equations with standard boundary conditions – this gives a concrete realization [27] of the celebrated ADHMN construction. What we will show is that D-strings in the near-horizon geometry of the D3-branes are controlled by the same equations, but with boundary conditions identical to those of Kronheimer's problem. This follows from the superconformal symmetry of the worldvolume theory. The vacuum configurations of N D-strings in AdS^5 are thus labelled again by N-dimensional representations ρ of SU(2). These supersymmetric 'fuzzy-sphere' bound states correspond in the holographic dual theory to heavy magnetic sources in non trivial representations of $SU(\infty)$.[3] The domain walls, which are the main theme of this paper, are kinks on the D-string worldsheet – they correspond to braiding operators of the Wilson-'t Hooft line in the dual CFT.

There is a number of related problems that we do not address. Our matrix domain walls also describe, for instance, supersymmetric instantons in the deformed matrix quantum mechanics of references [28, 29, 30], and may bear on the vacuum structure of M(atrix) theory and on scattering with longitudinal momentum transfer [31, 32]. They also occur as boundary RG flows between Cardy states of $SU(2)_k$ WZW models, much as in the Kondo problem. It may also be interesting to find a string realization of Kronheimer's original problem, that of $SO(3)$-invariant instantons in S^4. Finally, the behaviour of closed (circular or rectangular) Wilson-'t Hooft lines in higher representations of the gauge group is a very interesting problem that we do not address.

The organization of this paper is as follows. In section 2 we recall the vacuum structure of the $\mathcal{N} = 1^*$ theory and derive the equation as well as some obvious solutions for the classical BPS domain walls. In section 3, we discuss the relation to the standard monopole problem, and in particular the limit where a non-abelian gauge symmetry is restaured. Section 4 is an analysis of the existence of solutions

[3] We thank J. Maldacena for an early suggestion of this point.

interpolating between arbitrary representations, and relies a lot on Kronheimer's work. Section 5 considers the apparently unrelated problem of radial D-strings in $AdS_5 \times S^5$. We show that the transverse coordinates of the D-string satisfy conditions isomorphic to the $\mathcal{N} = 1^*$ domain walls, and discuss the holographically dual interpretation of the solutions. There is no harm in skipping the technicalities of section 4 in a first reading.

2. Supersymmetric domain walls in $\mathcal{N} = 1^*$ SYM

The $\mathcal{N} = 4$ Yang-Mills theory in four dimensions admits a deformation lifting all of its moduli space while preserving $\mathcal{N} = 1$ supersymmetry. This deformation is most easily described by rewriting the six real scalar fields as three chiral superfields Φ^a, taking their values in the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$ of the gauge group G. One can then deform the $\mathcal{N} = 4$ superpotential by a mass term,

$$W = \mathrm{Tr} \sum \left(\frac{1}{6} \, \epsilon^{abc} \Phi^a [\Phi^b, \Phi^c] - \frac{1}{2} m \; \Phi^a \Phi^a \right) \ , \tag{2.1}$$

which breaks the R-symmetry group from $SO(6)$ to $SO(3)$. We will use the same symbol for the superfield and for its scalar component, since the context will make clear what we mean. By redefining the phase of the fields, we can assume that m is real and positive. The D-term contribution to the scalar potential,

$$V = \frac{1}{g^2} \mathrm{Tr} \ D^2 \quad \text{with} \quad D = \sum [\Phi^a, (\Phi^a)^\dagger] \ . \tag{2.2}$$

is not affected by the mass deformation.

2.1 Vacua of mass deformed $\mathcal{N} = 4$ Yang-Mills

The supersymmetric classical vacua of the theory obey

$$\frac{\partial W}{\partial \Phi^a} = \frac{1}{2} \epsilon^{abc} [\Phi^b, \Phi^c] - m \Phi^a = 0 \ , \quad D = 0 \ , \tag{2.3}$$

and are hence in one-to-one correspondence with inequivalent embeddings of $SU(2)$ in $\mathcal{G}^{\mathbb{C}}$. For $G = SU(N)$ these are simply N-dimensional representations, ρ, of $SU(2)$. Such representations can be always unitarized, meaning that the three generators can be made antihermitean by a change of basis. The D-term conditions force this change of basis to be unitary, so that it can be undone by a gauge transformation. The vevs of the chiral fields in a vacuum ρ are thus given by

$$\Phi^a = m \rho^a \ , \quad [\rho^a, \rho^b] = \epsilon^{abc} \rho^c \ , \quad \rho^a = -(\rho^a)^\dagger \tag{2.4}$$

The representation ρ is in general reducible, and can be decomposed into different blocks of (integer or half-integer) spin j,

$$\rho \ = \ \bigoplus n_j \ [j] \qquad \text{with} \qquad \sum (2j+1) \ n_j = N \ . \tag{2.5}$$

The unbroken gauge symmetry in the corresponding vacuum is

$$G_{(\rho)} \equiv \left(\prod_j U(n_j) \right) \Big/ U(1) \ . \tag{2.6}$$

This contains in general abelian factors, which are asymptotically-free in the infrared. Vacua with a mass gap (and no abelian factors) correspond to the special representations which break up precisely into k identical blocks of size N/k. There is one such representation for each divisor k of N. The corresponding vacuum splits at the quantum level into k Higgs, confining and oblique vacua, related by the spontaneously broken R-symmetry.

For $G = SO(N)$ (or $USp(N)$), the embeddings of $SU(2)$ are again given by N-dimensional representations which must now be chosen real (respectively pseudo real). This implies that only integer (respectively half-integer) spins appear in the decomposition (2.5). Unbroken $U(1)$ gauge symmetries now arise whenever there is a pair of identical representations in the decomposition, i.e. for every j for which $n_j = 2$. As for embeddings of $SU(2)$ in the exceptional Lie groups, these have been classified in reference [33].

2.2 Supersymmetric domain walls

A theory with many isolated vacua is guaranteed, under some mild assumptions, to have smooth domain-wall solutions. For vacua with unbroken supersymmetry these walls may, but need not a priori, be supersymmetric. To see why let us review the argument leading to a BPS bound on the tension of domain walls in $\mathcal{N} = 1$ supersymmetric theories in four dimensions (see for instance [14]). The starting point is the energy functional for static configurations of the chiral fields,

$$E = \int \frac{d^3r}{g^2} \ \mathrm{Tr} \ \left(|\nabla\Phi^a|^2 + \left| \frac{\partial W}{\partial \Phi^a} \right|^2 + D^2 \right) \ . \tag{2.7}$$

Since we are interested in planar, $SO(1,2)$-invariant domain walls, we have set the gauge fields equal to zero. This is consistent as long as the Gauss conditions are satisfied,

$$\sum_a \ \left[\Phi^a, (\nabla\Phi^a)^\dagger\right] + \left[(\Phi^a)^\dagger, \nabla\Phi^a\right] = 0 \ . \tag{2.8}$$

The scalar fields are, furthermore, functions only of x, which is the coordinate parametrizing the transverse direction. The tension of the wall ($T \equiv E/\mathrm{Area}$) can then be written as a sum of squares plus a boundary term,

$$T = \int_{-\infty}^{+\infty} \frac{dx}{g^2} \ \left(\mathrm{Tr} \ \left| \left(\frac{d\Phi^a}{dx} \right)^\dagger - e^{i\alpha} \frac{\partial W}{\partial \Phi^a} \right|^2 + 2 \frac{d}{dx} \Re(e^{i\alpha} W) + \mathrm{Tr} \ D^2 \right) \ . \tag{2.9}$$

This expression leads to the lower (BPS) bound

$$T \geq \sup_\alpha \frac{2}{g^2} \Re(e^{i\alpha} W)\Big|_{-\infty}^{+\infty} = \frac{2}{g^2} |\Delta W| , \tag{2.10}$$

where α is any constant phase, and $\Re(A)$ stands for the real part of the quantity A.

The strictest bound is obtained when $e^{-i\alpha} = \Delta W / |\Delta W|$, and it is saturated by solutions of the first-order equations

$$\left(\frac{d\Phi^a}{dx}\right)^\dagger = e^{i\alpha} \frac{\partial W}{\partial \Phi^a} , \tag{2.11}$$

provided the D-terms, and the Gauss constraints also vanish. Note that the above equations imply

$$\frac{d}{dx} W(\Phi) = e^{i\alpha} \left|\frac{\partial W}{\partial \Phi^a}\right|^2 , \tag{2.12}$$

so that the superpotential 'moves' along a straight line, in the direction α, on the complex plane. Since in our case the superpotential at all the vacua is real, we may choose $e^{i\alpha} = -1$ (the choice $e^{i\alpha} = 1$ corresponds to flipping the sign of the coordinate x, which exchanges walls and anti-walls). The superpotential must thus be a real decreasing function from left to right. The BPS conditions (2.11) can, in fact, be interpreted as the equations of *gradient flow* for the potential $\Re(W)$. For a BPS-saturated wall to exist, gradient flow between two critical points of $\Re(W)$ must be allowed.

The $\mathcal{N} = 1^*$ superpotential evaluated at the vacuum ρ is proportional to the trace of the quadratic Casimir,

$$W(\rho) = m^3 \sum_j \frac{j(j+1)(2j+1)}{6} n_j . \tag{2.13}$$

For more general gauge groups G, the vacuum values of W are equal to $m^3/4$ times the Dynkin indices, $d(\rho)$, of the corresponding $SU(2)$ representations. A complete list of the $d(\rho)$ can be found in reference [33]. According to our previous discussion, BPS domain walls interpolating between ρ_- and ρ_+,

$$\Phi^a(x \to -\infty) \to m\rho^a_- \qquad \text{and} \qquad \Phi^a(x \to +\infty) \to m\rho^a_+ , \tag{2.14}$$

may exist only if $W(\rho_-) > W(\rho_+)$. Anti BPS domain walls interpolate, of course, in the opposite direction. Walls separating two vacua with $W(\rho_-) = W(\rho_+)$ are, on the other hand, necessarily *non-supersymmetric*. Such stable non BPS branes are generic when N is large, since there are exponentially many vacua and only polynomially many possible values for (2.13). Quantum corrections may lift this large degeneracy, but we will not pursue this question here further.

2.3 Some explicit solutions

A simple way of satisfying the D-term constraints (2.2) is by restricting the chiral fields Φ^a to be *antihermitean*. This is also the condition that would arise if we considered four-dimensional $N = 1$ Yang-Mills reduced to $D = 1$. The BPS equations then read

$$\frac{d\Phi^a}{dx} = \frac{1}{2}\epsilon^{abc}[\Phi^b, \Phi^c] - m\Phi^a \ . \tag{2.15}$$

As can be verified easily the Gauss conditions are then also automatically obeyed. Supersymmetric domain walls will thus exist whenever the equations (2.15) admit antihermitean solutions. The converse need not be, a priori, true – we will discuss this question later in section 4.4.

Some simple solutions of the above equations can be readily found. Using for instance the ansatz $\Phi^a = mf(mx)\ \rho_-^a$, leads to the differential equation $f\ ' = f(f-1)$. This can be integrated to give

$$\Phi^a(x) = \frac{m}{1+e^{m(x-x_0)}}\ \rho_-^a\ , \tag{2.16}$$

which is a domain wall interpolating between any initial vacuum ρ_- and the vacuum with unbroken gauge symmetry, $\rho_+ = [0]\oplus\cdots\oplus[0]$ (N times). A slight modification of this ansatz leads to solutions that interpolate between $\rho_- = \rho\otimes\tilde\rho$ and $\rho_+ = \tilde\rho\oplus\cdots\oplus\tilde\rho$ ($\dim\rho$ times), for any pair ρ and $\tilde\rho$ of representations. An explicit wall profile in this case is

$$\Phi^a(x) = \frac{m}{1+e^{m(x-x_0)}}\ \rho^a\otimes 1 + m\ 1\otimes\tilde\rho^{\ a}\ . \tag{2.17}$$

It can be checked that this solves the BPS equations, and obeys the appropriate boundary conditions. A particular example of this type is the wall that interpolates $[j-1/2]\oplus[j+1/2]\to[j]\oplus[j]$.

We do not know of a systematic method to construct explicit solutions of (2.15) in general. We will be able, nevertheless, to characterize their moduli spaces in section 4.

3. Nahm's equations and non-abelian monopoles

The form of (2.15) is reminiscent of the much studied Nahm equations, which give a dual description of $SU(2)$ monopoles on $\mathbb{R}^3$ [15, 16]. The only difference is the mass term, but this can be eliminated by the change of variables

$$\Phi^a \equiv -e^{-mx}X^a \qquad \text{and} \qquad s\equiv e^{-mx}/m\ , \tag{3.1}$$

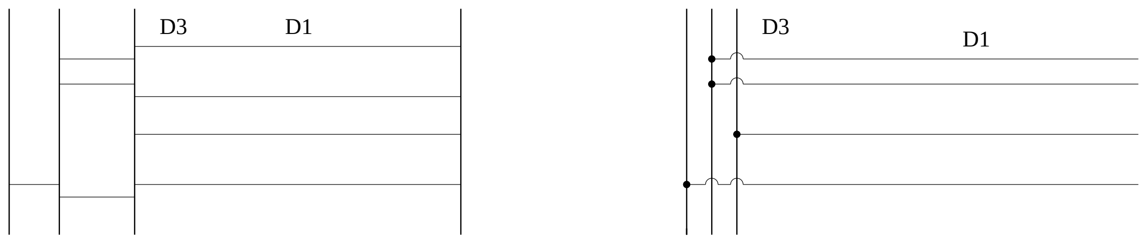


Figure 1: Left: D1/D3-brane configuration describing a (N_1,N_2,N_3) = (1,3,4) multimonopole of $SU(4)$. Right: D-brane configuration that could correspond to Nahm's equations on the semi-infinite line, with a reducible residue $\rho = [1/2]\oplus[0]\oplus[0]$ at the origin. The three D3-branes lie on top of each other.

which brings (2.15) to the standard Nahm form,

$$\frac{dX^a}{ds} = \frac{1}{2}\epsilon^{abc}[X^b, X^c] \ . \tag{3.2}$$

Note that s takes values on the semi-infinite interval $]0, +\infty[$, and the boundary conditions at both ends are

$$X^a(s \to 0) \sim -\frac{\rho_+^a}{s} + \text{finite} \ , \quad X^a(s \to \infty) \sim -\frac{\rho_-^a}{s} + \text{subleading} \ . \tag{3.3}$$

These differ from the boundary conditions in Nahm's description of the standard N-monopole problem [15, 16], where the X^a must have poles at both ends of a *finite* interval, with residues given by the *same irreducible* representation ρ of dimension N. In this section we would like to discuss this issue further.

Nahm's construction arises very naturally in a type IIB string-theory setting, where the monopoles are D-strings stretching between parallel D3-branes. A N-monopole corresponds to N D-strings whose coordinates are hermitean $N \times N$ matrices. What the Nahm equations describe is the evolution of these transverse matrix coordinates, as one moves along the D-string worldsheet [27, 34, 35]. We may visualize the D-strings as forming a 'fuzzy' spherical D3-brane, whose radius blows up at both ends of the finite s-interval, where the transverse coordinates have a pole. These blown up D-strings are in fact indistinguishable from the D3-branes on which they terminate [36], just as a D-string has a dual description as a spike or 'BIon' of the D3-brane [37, 38]. One may therefore forget the D3-branes altogether, and simply ask for a pole $X^a \sim \rho^a/(s - s_0)$ at the desired locations.

In the simplest case of $G = SU(2) \to U(1)$ there are precisely two poles, with residues controlled by the same irreducible representation ρ. This is consistent with the fact that the gauge theory can be engineered with just two D3-branes. In the general case $G = SU(n) \to U(1)^{n-1}$ the boundary conditions are more subtle [39], but are again easy to visualize from the D-brane perspective [34]. The novel feature is that the D-strings can now both *intersect and terminate* on the D3-branes (see

Figure 1). What one is instructed in the end to do is to solve $(n-1)$ Nahm equations for matrices of size $N_i \times N_i$, corresponding to N_i D-strings in the ith interval. The N_i are related in an obvious way to the monopole charges. The ith and $(i+1)$ solutions must be glued together by requiring a pole on the $p \times p$ block of the larger matrices (where $p = |N_i - N_{i+1}|$), and a step-function discontinuity on the remaining parts. This latter discontinuity is controlled by the open strings that stretch between the D-strings on each side of the D3-branes [34].

There are two limits in which the above description degenerates : (i) a D3-brane can move off to infinity, in which case some of the monopoles become infinitely heavy, or (ii) two or more D3-branes may coincide, in which case a non-abelian gauge symmetry is restaured and some of the monopoles become, formally at least, massless. Moving a D3-brane to infinity makes one of the s intervals semi-infinite. Moving, on the other hand, two or more D3-branes on top of each other forces two or more poles (corresponding, in general, to distinct irreducible representations of $SU(2)$) to collide. One is thus tempted, at first sight, to conclude that (3.3) are the appropriate boundary conditions in these limits.

The situation is, however, much more subtle. In a theory with a decoupled $U(1)$ the natural condition at infinity is that the X^a approach constant diagonal matrices, whose entries are the classical positions of the singular monopoles (see [40] for a detailed discussion). Our boundary condition is different: the X^a do approach zero at infinity, but it is crucial that the subleading behaviour be specified. As we will argue later, in section 5, this boundary condition arises naturally in the curved near-horizon geometry of D3-branes.

What about the boundary condition at $s \to 0$? If the residue ρ corresponds to a reducible representation of $SU(2)$ containing $\tilde{n} = \sum n_j$ irreducible blocks, then we are describing monopoles in a point of enhanced $SU(n \geq \tilde{n})$ gauge symmetry. This is indeed consistent with the fact that the D-strings grow into $\tilde{n}$ spherical D3-branes, whose sizes all diverge simultaneously, as illustrated in figure 1 (there can of course also exist extra D3-branes at this point). Whether one can assign magnetic monopoles to representations of an unbroken non-abelian gauge group has been, for many years, a subject of debate.[4] We dont have much to add to this debate here. We just note that there does exist a natural correspondence between residues ρ in the Nahm problem, and representations $r(\rho)$ of the unbroken gauge group. The Young tableau of the representation $r(\rho)$ has a row of $(2j+1)$ boxes for each spin j in the decomposition (2.5). This is illustrated in Figure 2.

The above assignement of a representation of the unbroken gauge group passes two simple consistency checks: (a) the number of boxes in the Young tableau is the same as the number N of D-strings – this is consistent with the expected n-

[4]We thank T. Tomaras for bringing this issue to our attention. For a recent discussion of the problem and for earlier references see [41] .

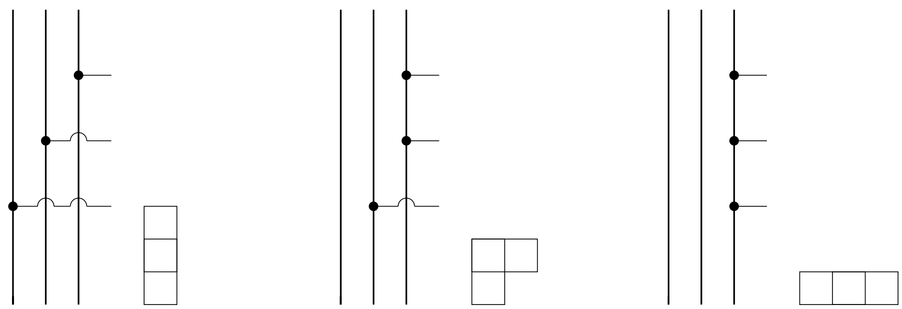


Figure 2: D1-branes configurations and associated Young tableaux as discussed in the text.

ality of the representation; (b) the number $\tilde{n}$ of rows is the minimal number of Chan Paton charges required to build a state of the representation $r(\rho)$, because of antisymmetrization. This fits nicely with the fact that $\tilde{n}$ is also the lower bound on the number of coincident D3 branes. We stress, however, that we have no clear reason to assign complete $SU(n)$ representations to our Nahm problem at present. We will come back to this question in the AdS/CFT context of section 5. The Young tableau of $r(\rho)$ is, in any case, a useful graphical representation of the $\mathcal{N}^* = 1$ vacuum corresponding to ρ, as will become apparent in section 4.3.

4. The moduli space of domain walls

Many of the techniques used in the analysis of the standard monopole problem can be carried over to the domain wall problem (2.15), or its mathematical equivalent (3.2) and (3.3) which describes, as we will see later, D-strings in AdS_5. The unusual boundary conditions introduce however some new twists: in particular, the problem still admits a Lax pair – but the spectral curve is degenerate, and the moduli space is still hyperkähler – but singular.

We are interested in solutions to (2.15) interpolating between two different vacua, characterized by two arbitrary unitary representations, ρ_- and ρ_+ of $SU(2)$. The embedding of these representations inside the gauge group G can be arbitrary. To render this explicit we thus write the boundary conditions as:

$$\lim_{x\to-\infty} \Phi^a = g\ \rho_-^a\ g^{-1}\ , \qquad \text{and} \qquad \lim_{x\to+\infty} \Phi^a = \rho_+^a\ , \tag{4.1}$$

where we fixed the embedding at $+\infty$, and allowed arbitrary rotations $g \in G$ at $-\infty$. Remarkably, this very problem has been studied by Kronheimer [18][5], in the context of $SO(3)$-invariant anti-self-dual connections on S^4: the superpotential (2.1) can indeed be thought as the Chern-Simons invariant of a left-invariant G-connection on S^3 specified by the three matrices Φ^a, and gradient flows of this functional yield

[5]We are grateful to O. Biquard and N. Hitchin for drawing our attention to this work.

anti-selfdual configurations on $\mathbb{R} \times S^3 \sim S^4$. We shall now review Kronheimer's results, perform some explicit computations of moduli spaces for particular choices of the pair (ρ_-, ρ_+), and draw conclusions as to the existence of BPS domain walls.

4.1 Lax pair, Moment map and Hyperkähler structure

In analogy with the usual Nahm problem, we can define in terms of the original variables Φ^a the matrices

$$L = (\Phi^3 + i\Phi^2) + 2i\zeta\Phi^1 + \zeta^2(\Phi^3 - i\Phi^2) \tag{4.2}$$

$$M = i\Phi^1 + \zeta(\Phi^3 - i\Phi^2) \ , \tag{4.3}$$

where ζ is the spectral parameter. Equation (2.15) can now be written as

$$e^{-i\alpha}\zeta^2 \frac{\partial}{\partial x}\bar{L}(-1/\bar{\zeta}) = [L(\zeta), M(\zeta)] - mL(\zeta) \ . \tag{4.4}$$

Imposing the reality condition $\zeta^2[L(-1/\bar{\zeta})]^\dagger = -L(\zeta)$, which corresponds to antihermitian matrices Φ, we find

$$\frac{\partial}{\partial x}L(x,\zeta) = [L, M] - mL \ . \tag{4.5}$$

This implies, in particular, the following first-order linear differential equations:

$$\frac{d}{dx}\mathrm{Tr}\ L^n = -mn\ \mathrm{Tr}\ L^n \tag{4.6}$$

for all $n > 0$. In order for the matrices to remain finite at $\pm\infty$, this requires $\mathrm{Tr}\ L^n = 0$ for all $n > 0$, or equivalently $L^N = 0$. This implies that the spectral curve $\det(L(\zeta) - \eta) = 0$ degenerates to $\eta^N = 0$. Our deformed Nahm problem can therefore be thought as a singular limit of the usual monopole problem when all constants of motion vanish. In particular, this implies that explicit solutions should be given in terms of hyperbolic functions, instead of the elliptic functions arising in the monopole problem. Indeed we have seen such an example in section 2.3, and it would be interested to generalize it to arbitrary pairs of vacua. Instead we shall study the moduli space of such domain walls, using techniques similar as in the standard monopole problem [42].

In order to make the hyperkähler structure manifest, let us introduce a gauge field Φ^0. The equation (2.15) becomes

$$\frac{d\Phi^a}{dx} + [\Phi^0, \Phi^a] = \frac{1}{2}\epsilon^{abc}[\Phi^b, \Phi^c] - m\Phi^a \tag{4.7}$$

and is invariant under the gauge transformations

$$\Phi^0 \to g\Phi^0 g^{-1} - \frac{dg}{dx}g^{-1} \ , \quad \Phi^a \to g\Phi^a g^{-1} \tag{4.8}$$

where g is an element of the (real) group G. The Nahm equations (4.7) can now be interpreted as the three moment maps for the action of the gauge group on the quaternionic vector space of matrices $(\Phi^0(x), \Phi^1(x), \Phi^2(x), \Phi^3(x))$ with metric

$$ds^2 = \int_0^\infty dx\, e^{mx}\ \mathrm{Tr}\ [(\Phi^0)^2 + (\Phi^1)^2 + (\Phi^2)^2 + (\Phi^3)^2]\ . \tag{4.9}$$

As usual, these equations can be split into a complex and a real one. Defining the complex variables

$$\alpha = \frac{1}{2}(\Phi^0 - i\Phi^1)\ , \qquad \beta = -\frac{1}{2}(\Phi^2 + i\Phi^3)\ , \tag{4.10}$$

we can express the equations (4.7) as follows:

$$\frac{d\beta}{dx} + m\beta + 2[\alpha, \beta] = 0 \tag{4.11}$$

$$\frac{d}{dx}(\alpha + \alpha^\dagger) + m(\alpha + \alpha^\dagger) + 2([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]) = 0\ . \tag{4.12}$$

The complex equation is now invariant under complex gauge transformations,

$$\alpha \to g\alpha g^{-1} - \frac{1}{2}\frac{dg}{dx}g^{-1}\ , \quad \beta \to g\beta g^{-1}\ . \tag{4.13}$$

Following the original approach of Donaldson [42], we can split the problem of solving the Nahm equations in two parts: (i) find all solutions of the complex equation with the required boundary conditions, modulo complex gauge transformations, and (ii) show that there exists a solution of the real equation in the complex conjugacy class of each solution of (i). Part (i) is purely topological, since all solutions to the complex equation (4.11) are locally pure gauge. Part (ii) can be proved to hold using variational techniques [18]. In the following we shall restrict ourselves to part (i), which already yields the moduli space with a particular choice of complex structure.

4.2 Nilpotent orbits and the moduli space of domain walls

Let us define

$$H_\pm = \rho_\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \quad X_\pm = \rho_\pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \quad Y_\pm = \rho_\pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ . \tag{4.14}$$

It may then be shown [18] that any solution of the complex equation (4.11) can be expressed as follows:

$$x \in]-\infty, 0] : \ \begin{cases} \alpha_1(x) = \frac{m}{4}H_- \\ \beta_1(x) = \frac{m}{2}Y_- \end{cases} \tag{4.15}$$

$$x \in [0, \infty[: \ \begin{cases} \alpha_2(x) = \frac{m}{4}H_+ \\ \beta_2(x) = \frac{m}{2}Y_+ \ + e^{-mx}\ e^{-mH_+x/2} \cdot Z \cdot e^{mH_+x/2}\ , \end{cases} \tag{4.16}$$

modulo a complex gauge transformation g_- (respectively g_+) approaching a constant (respectively the identity) when x goes to $-(+)\infty$. Here $Z \in \mathcal{Z}(\rho_+)$ is the centralizer of X_+ in $su(N)$. The essential idea behind this claim is that any α approaching $H_\pm$ at $\pm\infty$ is gauge equivalent to the constant $H_\pm$. The value of β is then obtained by solving the complex equation, and further adjusting the gauge transformation to come as close as possible to $\beta = mY_\pm/2$. Matching the two solutions (α_1, β_1) and (α_2, β_2) at $t = 0$ we find that

$$\frac{m}{2} Y_+ + Z \in \mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+) \ , \tag{4.17}$$

where $\mathcal{N}(\rho)$ denotes the set of elements of $\mathcal{G}^{\mathbb{C}}$ related to $Y(\rho)$ by conjugation under the complexified gauge group $G^{\mathbb{C}}$, and $\mathcal{S}(\rho)$ is the affine space $Y(\rho) + \mathcal{Z}(\rho)$. The element z so constructed is also unique, so that we arrive at Kronheimer's result: *the moduli space of solutions to Nahm's equations interpolating between $SU(2)$ representations ρ_- and ρ_+ is given by the intersection*

$$\mathcal{M}(\rho_-, \rho_+) = \mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+) \ . \tag{4.18}$$

In order to appreciate the significance of this result, it is useful to recall a number of properties of the spaces $\mathcal{N}(\rho)$ and $\mathcal{S}(\rho)$ [43, 18]:

(i) There is a one-to-one correspondence between the conjugacy classes of embeddings of $SU(2)$ into $\mathcal{G}$ and the $G^{\mathbb{C}}$-orbits of nilpotent elements in $\mathcal{G}^{\mathbb{C}}$. For $G = SU(N)$, this correspondence is provided by the Jordan canonical form. This justifies the notation $\mathcal{N}(\rho)$.

(ii) $\mathcal{S}(\rho)$ is a slice of $\mathcal{G}^{\mathbb{C}}$ transverse to the nilpotent orbit $\mathcal{N}(\rho)$, which it intersects only at the origin Y. In particular, for $\rho_- = \rho_+$, the moduli space reduces to a point Y, which describes the vacuum (no domain wall).

(iii) $\mathcal{S}(\rho_+)$ is the solution to a linear problem, namely the centralizer of the matrix X_+. On the other hand, $\mathcal{N}(\rho_-)$ is given by solutions to a set of polynomial equations (among which $s^n = 0$, where $n(\rho_-)$ is the order of nilpotency of ρ_-). The moduli space is therefore a ρ_--dependent algebraic variety in the space $\mathcal{S}(\rho_+)$.

(iv) For $\rho_+ = \mathbf{1}^N$ the trivial representation, we have $\mathcal{S}(\rho_+) = G^{\mathbb{C}}$, so that $\mathcal{M}(\rho_-, \rho_+) = \mathcal{N}(\rho_-)$: hence any representation ρ_- can be interpolated to the trivial representation through a solution of Nahm's equations. Indeed we have found an explicit example of such a solution in (2.16), but there is in fact a moduli space $\mathcal{N}(\rho_-)$ of them.

As an example of this construction, let us consider the simplest case of the domain wall interpolating between the irreducible representation $\rho_- = [N]$ of $SU(2)$ of dimension N (or more generally, the *principal* or *regular* embedding of $SU(2)$ into G) and $\rho_+ = [N-1]\oplus 1$ the *subregular* embedding. Since the subregular orbit has complex codimension 2 in the regular nilpotent orbit, the intersection $\mathcal{N}(\rho_+) \cap \mathcal{S}(\rho_-)$ has complex dimension 2. For G simply-laced, the moduli space is in fact [18] the

ALE space $\mathbb{C}^2/\Gamma$, where Γ is the discrete subgroup of $SU(2)$ of the same ADE type as the gauge group G. This is easily checked by explicit computation in the $G = SU(N)$ case: the matrices in $\mathcal{S}(\rho_-)$ form a $(N+2)$-dimensional subspace of $\mathbb{C}^{N^2}$ parameterized by[6]

$$s = \left(\begin{array}{cccccc|c} a_1 & a_2 & a_3 & \cdots & & a_{N-1} & b \\ 1 & a_1 & a_2 & \cdots & & a_{N-2} & 0 \\ 0 & 1 & a_1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & a_2 & \vdots \\ 0 & \cdots & 0 & 1 & & a_1 & 0 \\ \hline 0 & \cdots & \cdots & 0 & & c & d \end{array}\right) . \tag{4.19}$$

where all but one of the entries on the lower diagonal are fixed and we set $m = 2$. Requiring $z \in \mathcal{N}(\rho_+)$ is equivalent to imposing $s^N = 0$. This imposes the tracelessness condition $d = -(N-1)a_1$ and relates the off-diagonal values $a_i = \alpha_i a_1^i$ to the diagonal coefficient $a \equiv a_1$, where α_i are computable numerical coefficients. Finally, it imposes $bc = \alpha_N a^N$, which we recognize as the complex equation of the A_{N-1} singularity. This space is the singular limit of the well-known 4-dimensional gravitational instantons [44]. It would be interesting to understand under what circumstances the singularity might be resolved.

This strategy of parameterizing the linear space $\mathcal{S}(\rho_+)$ and imposing the nilpotency of z can be applied to any choice of representations (ρ_-, ρ_+), although it soon becomes rather cumbersome. Yet there is a simple way of deciding whether a BPS domain wall exists or not. Indeed, if such a wall exists, it will at least have a translational zero-mode, so that the moduli space $\mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)$ has to be non-empty. This condition is in fact equivalent to

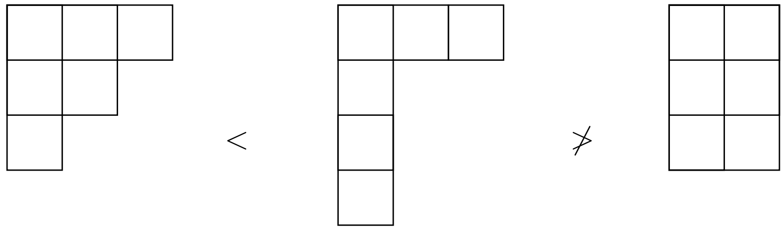
$$\mathcal{N}(\rho_+) \subset \overline{\mathcal{N}(\rho_-)} \; . \tag{4.20}$$

This gives a necessary and sufficient condition for the existence of BPS domain walls interpolating between two representations. This condition can be further explicited by noting that (for $SU(N)$) a nilpotent orbit (or a representation ρ) is uniquely labelled by the vector $k_p(\rho), p = 1 \dots N$ where $k_p(\rho)$ is the dimension of the null space of s^p, or equivalently the number of blocks in the Jordan decomposition of s^p. In particular, $k_1(\rho)$ is the number of irreducible blocks appearing in ρ, $k_N = N$ by definition, and

$$k_p(\rho) = 2n_2 + 3n_3 + 4n_4 + \cdots + (p-1)n_{p-1} + p(n_p + n_{p+1} + \cdots + n_N) \tag{4.21}$$

where the n_{2j+1} are the number of irreps of dimension $2j+1$ appearing in ρ, as in (2.5). All the $k_p(\rho)$ can only stay constant or increase as one goes from $\mathcal{N}(\rho)$ to its

[6]For the purpose of computing the dimension, we work in an non-normed basis, taking X in the Jordan form with coefficients 1 above the diagonal.



$\mathbf{3} \oplus \mathbf{2} \oplus \mathbf{1}$ $\mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ $\mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}$

Figure 3: Partial order between $SU(2)$ representation. The number of squares contained in the first p columns of the Young tableau has to increase for all p. The two representations on the right cannot be related by a BPS nor an anti-BPS domain wall.

closure, hence the condition (4.20) can be rewritten as

$$k_p(\rho_-) \leq k_p(\rho_+) \ , \quad p = 1 \dots N \tag{4.22}$$

This condition can be understood graphically in terms of Young tableaux associated to each reducible representation of $SU(2)$ as in Figure 3: $k_p(\rho)$ is the number of squares appearing in the first p columns of the Young tableau, and has to increase from ρ_- to ρ_+. In particular, the number of irreps (or fuzzy D$(p+2)$-branes in a more physical language) can only increase from $-\infty$ to $+\infty$, in addition to the decrease of the superpotential. It is also important to note that the order between $SU(2)$ representation is only partial, and there exists pairs of vacua with no BPS domain wall interpolating between them. This provides an interesting example of spontaneous supersymmetry breaking at a classical level.

4.3 Additivity rule and elementary domain walls

Let us start by discussing the structure of the affine space $\mathcal{S}(\rho)$, in the case of $G = SU(N)$. Taking for ρ the irreducible representation of dimension N, it is easy to see that the centralizer $Z(\rho)$ has dimension N: it consists of upper triangular matrices, with coefficients equal along the upper diagonals. If $\rho = \oplus_i \rho_i$ is reducible, the structure is similar in each $\dim(\rho_i) \times \dim(\rho_j)$ block, with $\min[\dim(\rho_i), \dim(\rho_j)]$ free coefficients in each block. Ordering the representations ρ_i by decreasing dimension and counting them with multiplicity[7], we find

$$\dim_{\mathbb{C}} \mathcal{S}(\rho) = \sum_i (2i-1) \dim(\rho_i) \ . \tag{4.23}$$

The dimension of $\mathcal{N}(\rho)$ is easily obtained from this result: the dimension of the $G^{\mathbb{C}}$ orbit of $Y(\rho)$ is equal to the dimension of $G^{\mathbb{C}}$ minus that of the centralizer of $Y(\rho)$, which is the same as that of the centralizer of $X(\rho)$. Hence

$$\dim_{\mathbb{C}} \mathcal{N}(\rho) = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} \mathcal{S}(\rho) \ . \tag{4.24}$$

[7]Note that the labelling of representations here is different from the one used in (2.5).

ρ	$\mathbf{6}$	$\mathbf{5}\oplus\mathbf{1}$	$\mathbf{4}\oplus\mathbf{2}$	$\mathbf{4}\oplus\mathbf{1}^2$	$\mathbf{3}^2$	$\mathbf{3}\oplus\mathbf{2}\oplus\mathbf{1}$	$\mathbf{3}\oplus\mathbf{1}^3$	$\mathbf{2}^3$	$\mathbf{2}^2\oplus\mathbf{1}^2$	$\mathbf{2}\oplus\mathbf{1}^4$	$\mathbf{1}^6$
$n(\rho)$	6	5	4	4	3	3	3	2	2	2	1
$\mathcal{D}(\rho)$	35	20	11	10	8	5	4	3	2	1	0
$\dim_{\mathbb{C}} \mathcal{N}(\rho)$	30	28	26	24	24	22	18	18	16	10	0
$\dim_{\mathbb{C}} \mathcal{S}(\rho)$	6	8	10	12	12	14	18	18	20	26	36
$n_>(\rho)$	48	52	56	60	60	64	72	72	76	88	108
$n_0(\rho)$	35	34	34	31	32	33	26	27	28	19	0
$n_<(\rho)$	25	22	18	17	16	11	10	9	4	1	0
$n^{\mathbb{C}}_>(\rho)$	73	74	74	77	76	75	82	81	80	89	108

Table 1: Order of nilpotency $n(\rho)$, Dynkin index $\mathcal{D}(\rho)$, dimension of the nilpotent orbit $\mathcal{N}(\rho)$ and of the shifted centralizer $\mathcal{S}(\rho)$ for six-dimensional representations ρ of $SU(2)$, as in section 4.2. The last four rows give the number of positive (resp. zero and negative) eigenvalues of the Hessian of W at the critical point ρ, as in section 4.4.

The dimension of the moduli space (4.18) can now be computed as follows.[8] Assuming that $\mathcal{N}(\rho_+) \subset \overline{\mathcal{N}(\rho_-)}$, the intersection $\mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)$ is non-empty, so that the sum $\mathcal{N}(\rho_-) + \mathcal{S}(\rho_+)$ is well-defined. We then have

$$\dim_{\mathbb{C}}(\mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)) = \dim_{\mathbb{C}} \mathcal{N}(\rho_-) + \dim_{\mathbb{C}} \mathcal{S}(\rho_+) - \dim_{\mathbb{C}}(\mathcal{N}(\rho_-) + \mathcal{S}(\rho_+)) \ .$$

Since $\mathcal{N}(\rho_+) + \mathcal{S}(\rho_+)$ generates all of $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{N}(\rho_+) \subset \overline{\mathcal{N}(\rho_-)}$, the last term in the equation above is equal to $\dim_{\mathbb{C}} G$. Using (4.24), we arrive at

$$d(\rho_-, \rho_+) \equiv \dim_{\mathbb{C}} \mathcal{M}(\rho_-, \rho_+) = \dim_{\mathbb{C}} \mathcal{S}(\rho_+) - \dim_{\mathbb{C}} \mathcal{S}(\rho_-) \qquad (4.25)$$

where $\dim_{\mathbb{C}} \mathcal{S}(\rho_)$ can be computed using (4.23). We have tabulated in Table 1 the Dynkin index and dimensions of nilpotent orbits and centralizer for $N = 6$, together with some further data to be discussed in the next section. Table 2 gives the dimension of the moduli space for low values of N, computed using (4.25).

These tables call for a number of observations.

(i) All spaces have an even complex dimension, as required by the hyperkähler property. A vanishing dimension means that the domain wall does not exist, since the translational zero-mode is always present.

(i) All spaces have an even complex dimension, as required by the hyperkähler property. A vanishing dimension means that the domain wall does not exist, since the translational zero-mode is always present.

(ii) All allowed domain walls satisfy $W(\rho_-) > W(\rho_+)$ as well as the stronger condition (4.22), even though this criterium has not been used in deriving them.

[8] We thank P. Slodowy for suggesting this route to Equation (4.25). The same result will be obtained in the next section using Morse theory arguments.

$\rho_-\backslash\rho_+$	$\mathbf{2}$	$\mathbf{1}^2$
$\mathbf{2}$	0	$\mathbb{C}^2/\mathbb{Z}_2$
$\mathbf{1}^2$		0

$\rho_-\backslash\rho_+$	$\mathbf{3}$	$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{1}^3$
$\mathbf{3}$	0	$\mathbb{C}^2/\mathbb{Z}_3$	6
$\mathbf{2}\oplus\mathbf{1}$		0	4
$\mathbf{1}^3$			0

$\rho_-\backslash\rho_+$	$\mathbf{4}$	$\mathbf{3}\oplus\mathbf{1}$	$\mathbf{2}^2$	$\mathbf{2}\oplus\mathbf{1}^2$	$\mathbf{1}^4$
$\mathbf{4}$	0	$\mathbb{C}^2/\mathbb{Z}_4$	4	6	12
$\mathbf{3}\oplus\mathbf{1}$		0	$\mathbb{C}^2/\mathbb{Z}_2$	4	10
$\mathbf{2}^2$			0	$\mathbb{C}^2/\mathbb{Z}_2$	8
$\mathbf{2}\oplus\mathbf{1}^2$				0	6
$\mathbf{1}^4$					0

$\rho_-\backslash\rho_+$	$\mathbf{5}$	$\mathbf{4}\oplus\mathbf{1}$	$\mathbf{3}\oplus\mathbf{2}$	$\mathbf{3}\oplus\mathbf{1}^2$	$\mathbf{2}^2\oplus\mathbf{1}$	$\mathbf{2}\oplus\mathbf{1}^3$	$\mathbf{1}^5$
$\mathbf{5}$	0	$\mathbb{C}^2/\mathbb{Z}_5$	4	6	8	12	20
$\mathbf{4}\oplus\mathbf{1}$		0	$xy^4 = z^3$	4	6	10	18
$\mathbf{3}\oplus\mathbf{2}$			0	$\mathbb{C}^2/\mathbb{Z}_2$	4	8	16
$\mathbf{3}\oplus\mathbf{1}^2$				0	$\mathbb{C}^2/\mathbb{Z}_2$	6	14
$\mathbf{2}^2\oplus\mathbf{1}$					0	4	12
$\mathbf{2}\oplus\mathbf{1}^3$						0	8
$\mathbf{1}^5$							0

Table 2: Moduli spaces for domain walls in $SU(N)$, $N \leq 6$. The entry denotes the complex dimension of the moduli space, or the space itself if of dimension 2.

Starting from $N = 6$, we find cases such as $\mathbf{4}\oplus\mathbf{1}^2 \to \mathbf{3}^2$ or $\mathbf{3}\oplus\mathbf{1}^3 \to \mathbf{2}^3$ where no BPS domain wall exists, even though the superpotential does decrease. Similarly, from $N = 8$ on we find pairs of vacua with same superpotential such as $\mathbf{3} \oplus \mathbf{1}^5$ and $\mathbf{2}^4$, for which the same criterium excludes the existence of tensionless domain walls. Of course, non-BPS domain walls with these boundary conditions do exist, so that supersymmetry is spontaneously broken. on the basis of the superpotential alone.

(iii) A number of spaces have complex dimension 2 and correspond to unresolved ALE spaces; the case $\mathbf{4}\oplus\mathbf{1} \to \mathbf{3}\oplus\mathbf{2}$ is particularly interesting since it exhibits a non-isolated singularity.

(iv) The dimensions of the moduli spaces satisfy the additivity rule

$$d(\rho_-, \rho_+) = d(\rho_-, \rho_0) + d(\rho_0, \rho_+) \ , \tag{4.26}$$

as follows from (4.25). This additivity rule suggests that all domain walls can be seen as composite of elementary domain walls, for which $W(\rho_-) - W(\rho_+)$ is minimal (but

$\rho_-\backslash\rho_+$	$\mathbf{6}$	$\mathbf{5}\oplus\mathbf{1}$	$\mathbf{4}\oplus\mathbf{2}$	$\mathbf{4}\oplus\mathbf{1}^2$	$\mathbf{3}^2$	$\mathbf{3}\oplus\mathbf{2}\oplus\mathbf{1}$	$\mathbf{3}\oplus\mathbf{1}^3$	$\mathbf{2}^3$	$\mathbf{2}^2\oplus\mathbf{1}^2$	$\mathbf{2}\oplus\mathbf{1}^4$	$\mathbf{1}^6$
$\mathbf{6}$	0	$\mathbb{C}^2/\mathbb{Z}_6$	4	6	6	8	12	12	14	20	30
$\mathbf{5}\oplus\mathbf{1}$		0	2	4	4	6	10	10	12	18	28
$\mathbf{4}\oplus\mathbf{2}$			0	$\mathbb{C}^2/\mathbb{Z}_2$	$\mathbb{C}^2/\mathbb{Z}_2$	4	8	8	10	16	26
$\mathbf{4}\oplus\mathbf{1}^2$				0	0	2	6	6	8	14	24
$\mathbf{3}^2$				0	0	2	6	6	8	14	24
$\mathbf{3}\oplus\mathbf{2}\oplus\mathbf{1}$						0	4	4	6	12	22
$\mathbf{3}\oplus\mathbf{1}^3$							0	0	2	8	18
$\mathbf{2}^3$							0	0	2	8	18
$\mathbf{2}^2\oplus\mathbf{1}^2$									0	6	16
$\mathbf{2}\oplus\mathbf{1}^4$										0	10
$\mathbf{1}^6$											0

Table 2 (*continued*)

strictly positive) at fixed ρ_+: the latter appear above the diagonal in Tables 2. It is thus sufficient to understand the moduli of the elementary domain walls. To this end, note that besides translations there are two other classes of continuous deformations of the solutions, (a) $\Phi^a(t) \to g\Phi^a(t)g^\dagger$, where g is a global transformation in the gauge group G; and (b) $\Phi^a(t) \to \rho_+(R)(R^a{}_b\Phi^b(t))\rho_+(R)^\dagger$, with R an $SU(2)$ rotation. Deformations (a) preserve the boundary conditions (4.1) iff g is in the centralizer of ρ_+. Deformations (b) preserve the boundary conditions for arbitrary R; they act non-trivially except for a finite subgroup Γ of SU(2), since (b) acts at $t = -\infty$ as a transformation (a) with $g = \rho_+(R)\rho_-(R^{-1})$ and ρ_+ and ρ_- are different representations. Hence (b) gives an isometric action of $SU(2)$ on the moduli space with three-dimensional orbits [18]. The $SU(2)$ action plus translations accounts for the moduli of the $[n+1] \to [n] \oplus \mathbf{1}$ walls, as well as all elementary walls of complex dimension two in tables 2 (elementary walls correspond to entries just above the diagonal). For those cases, the actions of (a) and (b) can be shown to be equivalent. The remaining elementary walls have moduli beyond those of (b) which can be all understood in terms of (a). For example the $\mathbf{2} \oplus \mathbf{1}^{N-2} \to \mathbf{1}^N$ walls (last non-zero entry on the lower right corner of the tables) have one translation, and N^2-1 group conjugations out of which those corresponding to the subgroup $U(N-2)$ act trivially. This makes a total of $4(2N-1)$ real moduli, in agreement with the results of the tables. Note that transformations (a) and (b) are not independent, as can be verified easily in some examples. Hence the elementary walls have no moduli other than translations, plus global G and R-symmetry rotations. The 'additivity' rule furthermore shows that the moduli of composite walls can be also accounted for by these operations.

4.4 Morse theory and dimension of moduli space

We now would like to rederive the dimension of the moduli space $\mathcal{M}(\rho_-, \rho_+)$, obtained in (4.25), in a different way. We will assuming that the space is non-empty, and then use simple arguments about solutions of ordinary differential equations. One advantage of this more simple-minded approach is that it shows the robustness of the result under small deformations of the Kähler potential.

Assume then that there exists a solution of a gradient flow in $\mathbb{R}^N$,

$$\frac{d\phi^i}{dt} = -g^{ij}\frac{\partial W}{\partial \phi^j} \ , \tag{4.27}$$

interpolating between two critical points ρ_- and ρ_+ of the (real) potential Wat $\pm\infty$. We now make use of the following fact: *the number of zero modes around solutions flowing between two different vacua of W is given generically by the difference of the number of strictly positive eigenvalues of the Hessian of W at both ends of the flow.* The reason for this is illustrated on Fig. 4: let us choose a small sphere S^{N-1} around the point ρ_- at $-\infty$, and choose the initial condition ϕ on that sphere close to the trajectory we wish to perturb. As we evolve backward in time, the solution will not reach ρ_- unless ϕ is on the separatrix between strictly positive and negative eigenvalues of the Hessian of W at ρ_-: this puts $n_>(\rho_-)$ conditions on ϕ on S^{N-1}, where $n_>(\rho)$ is the number of strictly positive eigenvalues of the Hessian at the critical point ρ. Now let us evolve the solution forward in time until we reach the vicinity of ρ_+: similarly, the solution will not attain the critical point at ρ_+ unless it arrives along the separatrix, which puts $n_\leq(\rho_+)$ conditions on p. Altogether, the dimension of the space of solutions of (2.11) interpolating between $\rho_\pm$ at $\pm\infty$ and close to the solution of reference reads[9]

$$d(\rho_-, \rho_+) = (N-1) - n_>(\rho_-) - n_\leq(\rho_+) + 1 = n_>(\rho_+) - n_>(\rho_-) \tag{4.28}$$

where we added in the translational zero-mode. This is in agreement with the intuitive fact that the number of positive eigenvalues of the Hessian should increase from one fixed point to another along the gradient flow (4.27). Note also that the statement in (4.28) applies only to generic trajectories, for which the conditions we imposed at $\pm\infty$ are independent. For degenerate cases, the r.h.s. of (4.28) gives only a *lower* bound on the dimension of the moduli space.

In order to compute the dimension, we thus need to study the eigenvalues of the Hessian of W at a critical point (2.4), given by the operator

$$\mathcal{H}_\rho : \quad K_a \to -\epsilon^{abc}[m\rho^b, K^c] + mK_a \tag{4.29}$$

[9] In line with (4.1), we have assumed that the boundary conditions require ϕ^i to be *at* the critical point ρ_+ at $+\infty$, and *on* the critical locus ρ_- at $-\infty$. This fixes the zero-modes of the solution at $+\infty$.

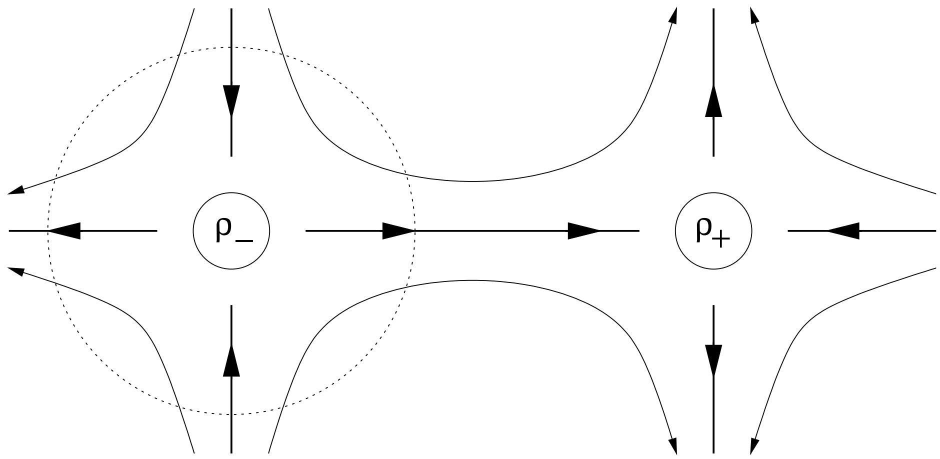


Figure 4: Gradient flow between two vacua.

acting on a triplet of antihermitian $N \times N$ matrices K_a. We note that the spectrum of this operator is simply obtained from that of

$$\mathcal{J}_\rho : \quad k^a \to \epsilon^{abc} \rho^b k^c \ , \tag{4.30}$$

through

$$Sp(\mathcal{H}_\rho) = m \left[1 - Sp(\mathcal{J}_{\rho\otimes\rho})\right] \ . \tag{4.31}$$

$\mathcal{J}_{\rho\otimes\rho}$ acts in the same vector space as $\mathcal{H}_\rho$ but now seen as a space of triplet of N^2-dimensional vectors. The spectrum of $\mathcal{J}$ is easily computed on irreducible representations, since $\mathcal{J}_\rho = J^a_\rho \otimes J^a_{\mathbf{[3]}}$,

$$Sp(\mathcal{J}_{\mathbf{[2j+1]}}) = \{(j+1)|_{2j-1},\ 1|_{2j+1},\ -j|_{2j+3}\} \quad (j>0)\ ,\ Sp(\mathcal{J}_{\mathbf{[1]}}) = \{0|_3\}\ , \tag{4.32}$$

where we denoted the multiplicity in subscript, and in arbitrary representations using the sum rule

$$Sp(\mathcal{J}_{\rho\oplus\rho'}) = Sp(\mathcal{J}_\rho) \cup Sp(\mathcal{J}_{\rho'}) \ . \tag{4.33}$$

We can thus compute the spectrum of $\mathcal{H}_\rho$ by reducing the tensor product $\rho\otimes\rho$ into irreductible components and using (4.32). In particular, it is easy to see that the number of zero eigenvalues of $\mathcal{H}_\rho$ is $n_0(\rho) = N^2 - \sum n_i^2$, which is also the number of broken generators in the vacuum (2.4): the only flat directions of the superpotential are therefore gauge rotations as expected. The number of strictly positive eigenvalues can also be computed, and is given by

$$n_>(\rho) = N^2 + 2\sum_i (2i-1)\dim(\rho_i) \tag{4.34}$$

where the ρ_i are the irreducible blocks appearing in the representation ρ, ordered by decreasing dimension as in (4.23). Using (4.28), we have thus reproduced the dimension formula (4.25). This confirms that the equality in (4.28) holds for any choice of ordered representations ρ_-, ρ_+. Furthermore, it shows that the dimension formula is given by an index, and hence is robust under small deformations of the Kähler potential (*i.e* the metric g^{ij} appearing in (4.27)).

The number of strictly negative eigenvalues, or Morse index, is also interesting, since it yields the fermion number of the vacuum ρ in the supersymmetric quantum mechanics. Its parity is the same as that of $N - \sum n_i$, so that bosonic vacua are those with N irreductible components modulo 2. The number of positive, zero and negative eigenvalues for $N = 6$ are displayed in the lower part of Table 1.

Before proceeding further let us comment on the original problem (2.11), where the matrices Φ^a were not assumed to be antihermitian but general complex matrices. The complex flow (2.11) is equivalent to a gradient flow for the real and imaginary parts of the Φ^a, with a potential $\Re(W)$. Since the potential $\Re(W)$ is harmonic, the numbers of (strictly) positive and negative eigenvalues are the same. They are related to the ones in the real case as follows:

$$n_>^{\mathbb{C}}(\rho) = n_<^{\mathbb{C}}(\rho) = n_>(\rho) + n_<(\rho) \ , \quad n_0^{\mathbb{C}}(\rho) = 2 n_0(\rho) \ . \tag{4.35}$$

Assuming that the dimension formula still applies, we find a dimension smaller than the one for the real problem, as one can see by comparing the first and last rows in Table 1. Such a conclusion must be clearly wrong, since the solutions of the real problem are also solutions of the complex problem. This signals that the restrictions imposed at $\pm\infty$ are not independent in the complex case, and gives us confidence that the complex problem does not have any extra solutions. We have not been able to prove this statement, however.

5. D strings in $AdS_5 \times S^5$ and holography

We return now to the magnetic monopole problem discussed in section 3. What we will show is that the vacua and domain walls of the $\mathcal{N}^* = 1$ theory, arise also when one studies N D-strings in the near horizon geometry of D3-branes. A single D-string stretching radially outwards in AdS_5 is a magnetic source in the fundamental representation of the dual $SU(n)$ gauge theory [45, 46]. Several coincident D-strings in 'fuzzy sphere' configurations must be dual to Wilson-'t Hooft lines in higher representations of $SU(n)$, as we will here try to argue.

The metric and the Ramond-Ramond background in AdS_5 read:

$$ds^2 = L^2 \, \frac{dy^2 + dx_\mu dx^\mu}{y^2} \ , \quad \text{and} \quad C_{y0ab} = L^2 \, \frac{\epsilon_{abc} x^c}{y^5} \ . \tag{5.1}$$

Here (y, x^μ) are the usual Poincaré coordinates, with $x^\mu = (x^0, x^a)$ parameterizing the worldvolume of the background D3 branes. We chose a convenient gauge for the antisymmetric four-form potential, whose field strength ($H = dC$) must be proportional to the volume form of AdS_5. The coordinate y is the inverse radial distance from the D3-branes, and $y = 0$ is the AdS boundary.

Consider now N D-strings stretching radially outwards from the D3-branes. Their worldvolume theory contains (non-abelian) Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) terms, which combine to give an energy functional

$$E = T_D L^2 \int_0^\infty \frac{dy}{2y^6} \, \mathrm{Tr} \, \left(y^2 \frac{\partial X^a}{\partial y} - \frac{1}{2} \epsilon^{abc} [X^b, X^c] \right)^2 + E_{\mathrm{boundary}} + \cdots \tag{5.2}$$

We have here used y and $t = x^0$ to parametrize the D-string worldvolume. The dots include higher-order terms, which can be neglected thanks to the supersymmetry of the problem, as well as the rest mass of the stretched strings which is a constant. E_{boundary} is a boundary contribution to the energy which we will determine shortly. We have assumed a static configuration, and have also set the worldvolume gauge field and the S^5 coordinates to zero. Finally T_D is the D-string tension.

The various terms in the expression (5.2) can be understood as follows: The gradient square and commutator square terms are the usual lowest-order contributions from the DBI action. The powers of y that accompany them can be guessed from the scale invariance ($y \to \lambda y, \; X^\mu \to \lambda X^\mu$) of the problem. The cross term in the expansion of the square would have been a total derivative in flat space-time. Here it contributes a bulk term which (after integration by parts) can be recognized as the non-Abelian WZ coupling to the Ramond-Ramond four-form proposed by Myers [20]. The fact that the energy density is a perfect square is of course a consequence of the unbroken supersymmetries of the background. Indeed, one could have used this argument to discover the non-Abelian WZ coupling.

The supersymmetric configurations of D-strings are given by solutions to the equations

$$y^2 \frac{dX^a}{dy} = \frac{1}{2} \epsilon^{abc} [X^b, X^c] \; . \tag{5.3}$$

Furthermore the boundary conditions that respect scale invariance are

$$X^a(y \to 0) = -y\rho_-^a + \text{subleading} \; , \quad X^a(y \to \infty) = -y\rho_+^a + \text{finite} \; . \tag{5.4}$$

By changing variables to $s = 1/y$, we recognize immediately the mathematical problem analyzed in the previous sections. The 'vacuum states' of the D-strings correspond to the trivial solutions

$$X^a = -y\rho^a \;\; \text{for all} \;\; y \; . \tag{5.5}$$

They correspond to 'fuzzy-sphere' bound states, described by the SU(2) representation ρ. Domain walls give transitions between these different vacua, as one moves from the ultraviolet ($s = \infty$) towards the infrared ($s = 0$).

Because of the $\mathcal{N} = 4$ supersymmetry, the mass of the bound states should only depend on the total number, N, of D-strings. We may use this argument to show that

$$E_{\text{boundary}} = 0 \ . \tag{5.6}$$

Indeed, the candidate boundary contribution to the energy is $\sim W(X)/y^4$, which is the difference between the perfect square terms and the DBI plus WZ couplings. If such a term were really present there would be a ρ-dependent contribution to the D-string mass (proportional to the ultraviolet cutoff δy). Since this is forbidden by supersymmetry, we conclude that, contrary to the situation in flat spacetime, such a term is not present here. This is analogous to the argument given for fundamental strings in reference [47].

Following our discussion in section 3, we expect a fuzzy-sphere configuration ρ of the D-strings to be dual to a Wilson-'t Hooft line in a higher representation of the gauge group. There are two natural candidates for this latter: (a) the irreducible representation $r(\rho)$ of section 3, or (b) the reducible representation corresponding to a Wilson-'t Hooft loop

$$\prod_j \mathrm{tr}^{n_j} \ (U^{2j+1}) \ , \tag{5.7}$$

where trU corresponds to a single D-string, in the fundamental representation of $SU(n)$. It is unclear to us which of the two interpretations is correct. The kinks on the D-string worldvolume should be, in any case, dual to topological twist operators acting on the above Wilson-'t Hooft lines. Since $W(\rho_-) > W(\rho_+)$ always, the ultraviolet to infrared flow tends to effectively increase the number of spherical 'fuzzy' D3 branes, in accordance with the naive entropy expectation. It would be interesting to associate the superpotential W to an entropy. We hope to return to these questions in some future work.

Acknowledgments

The authors are grateful to V. Bach, V.J. Balasubramanian, S. Cherkis, M. Douglas, P. Fayet, N. Hitchin, C. Houghton, J. Hübschmann, B. Kol, J. Maldacena, W. Nahm, N. Nekrasov, A. Smilga, C. Sonnenschein, P. Slodowy, T. Tomaras, D. Tong, C. Vafa and S. T. Yau for useful discussions and remarks. The work of B. P. is supported in part by the David and Lucile Packard foundation. C. B. and J. H. would like to thank Harvard University for its hospitality during the early stages of this work.

Finally, we would like to acknowledge the support of the European networks FMRX-CT96-0012, FMRX-CT96-0045, and HPRN-CT-2000-00122. Shortly after our work was released, a paper appeared by A. Frey [48] who addresses the gravity description of the $\mathcal{N} = 1^*$ domain walls. We thank him for pointing out a counter-example to the existence conjecture we suggested in the first version of our work. The conjecture has been reformulated and established in the present version.

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